

SIMPLICIAL AND CATEGORICAL COMMA CATEGORIES

STEVEN R. COSTENOBLE

ABSTRACT. We consider four categories: the category of diagrams of small categories indexed by a given small category \mathcal{O} , the (comma) category of small categories over \mathcal{O} , the category of diagrams of simplicial sets indexed by \mathcal{O} , and the category of simplicial sets over the nerve of \mathcal{O} . Fritsch and Golasinski claimed that these four categories have equivalent homotopy categories but, in fact, their proof contains an error and the homotopy categories are not equivalent with the weak equivalences they use in the comma categories. We show here that the correct weak equivalences are the “weak fibre homotopy equivalences” defined by Latch. We also construct a model category structure on the category of simplicial sets over $N\mathcal{O}$ in which the weak equivalences are the weak fibre homotopy equivalences.

1. INTRODUCTION

Let \mathbf{Cat} denote the category of small categories, let \mathcal{S} denote the category of simplicial sets, and let \mathcal{O} be a small category. In [FG98], Fritsch and Golasinski considered the relationship between four categories: $\mathcal{O}\text{-Cat}$, the category of contravariant functors $\mathcal{O} \rightarrow \mathbf{Cat}$, $\mathbf{Cat} \downarrow \mathcal{O}$, the category of small categories over \mathcal{O} , $\mathcal{O}\text{-}\mathcal{S}$, the category of contravariant functors $\mathcal{O} \rightarrow \mathcal{S}$, and $\mathcal{S} \downarrow N\mathcal{O}$, the category of simplicial sets over the nerve $N\mathcal{O}$. These categories fit into the following diagram.

$$\begin{array}{ccc} \mathbf{Cat} \downarrow \mathcal{O} & \begin{array}{c} \xleftarrow{E} \\ \xrightarrow{F} \end{array} & \mathcal{O}\text{-Cat} \\ N \downarrow & & \downarrow N \\ \mathcal{S} \downarrow N\mathcal{O} & \begin{array}{c} \xleftarrow{E} \\ \xrightarrow{F} \end{array} & \mathcal{O}\text{-}\mathcal{S} \end{array}$$

In this diagram, the maps labeled F are left adjoints of the maps labeled E , the maps labeled N are induced in the obvious way by the nerve functor, and $FN = NF$.

The categories $\mathcal{O}\text{-}\mathcal{S}$ and $\mathcal{O}\text{-Cat}$ arise naturally in equivariant algebraic topology as models for the homotopy theory of G -spaces when G is a discrete group and \mathcal{O} is the category of orbits of G . In that context, the appropriate model category structure on $\mathcal{O}\text{-}\mathcal{S}$ is the one in which a natural transformation $\phi \rightarrow \psi$ is a fibration or a weak equivalence exactly when it is an objectwise fibration or weak equivalence, respectively. Similarly, the appropriate weak equivalences in $\mathcal{O}\text{-Cat}$ are the objectwise weak

Date: November 29, 2000.

1991 *Mathematics Subject Classification.* Primary 55U10; Secondary 18A25, 18G30, 18G55, 55P15, 55P91, 55U35.

equivalences (where weak equivalence of small categories is weak equivalence of their nerves); we can define a model category structure on $\mathcal{O}\text{-Cat}$ in which the fibrations are objectwise fibrations, using the model category structure given by Thomason [Tho80]. Fritsch and Golasinski claim that the adjunctions above are Quillen equivalences when we use on $\text{Cat}\downarrow\mathcal{O}$ and $\mathcal{S}\downarrow N\mathcal{O}$ the model category structures in which a map over \mathcal{O} or $N\mathcal{O}$ is a cofibration, fibration, or weak equivalence exactly when it is so considered as an ordinary map. Unfortunately, their proof contains an error, and their claim is not true, as shown by simple examples (see Example 2.1).

The correct weak equivalences to use in $\text{Cat}\downarrow\mathcal{O}$ and $\mathcal{S}\downarrow N\mathcal{O}$ are the *weak fibre homotopy equivalences* defined in [Lat79]. In Section 3 we give a correct proof of the equivalences of the homotopy categories of $\text{Cat}\downarrow\mathcal{O}$ and $\mathcal{O}\text{-Cat}$ and of $\mathcal{S}\downarrow N\mathcal{O}$ and $\mathcal{O}\text{-}\mathcal{S}$. This gives another proof of the equivalence of the homotopy categories of $\text{Cat}\downarrow\mathcal{O}$ and $\mathcal{S}\downarrow N\mathcal{O}$ shown in [Lat79]. In the remainder of the paper we construct a model category structure on $\mathcal{S}\downarrow N\mathcal{O}$ having weak fibre homotopy equivalences as its weak equivalences. Defining a model category structure on $\text{Cat}\downarrow\mathcal{O}$ appears to be much more difficult and we shall not attempt it here.

2. THE ADJOINT FUNCTORS

In this section we define and examine the adjoint pairs $(F, E): \text{Cat}\downarrow\mathcal{O} \rightarrow \mathcal{O}\text{-Cat}$ and $(F, E): \mathcal{S}\downarrow N\mathcal{O} \rightarrow \mathcal{O}\text{-}\mathcal{S}$.

First, let us recall some standard notations. Let Δ be the category whose objects are the ordered sets $[n] = \{0, 1, \dots, n\}$, $n \geq 0$, and whose maps are the order-preserving functions. We often think of $[n]$ as a category in the usual way, so that the maps in Δ can be thought of as functors. We write $\Delta[n]$ for the nerve of $[n]$, the standard n -simplex. We write $\partial\Delta[n] \subset \Delta[n]$ for the subsimplicial set generated by all of the $(n-1)$ -dimensional faces of $\Delta[n]$. If $0 \leq k \leq n$, we write $\Lambda^k[n] \subset \Delta[n]$ for the k th *horn*, the subsimplicial set generated by all of the $(n-1)$ -dimensional faces of $\Delta[n]$ except the one opposite the k th vertex.

Now, we define $F: \text{Cat}\downarrow\mathcal{O} \rightarrow \mathcal{O}\text{-Cat}$ as follows. If $\phi: \mathcal{C} \rightarrow \mathcal{O}$ is a small category over \mathcal{O} and b is an object of \mathcal{O} , we let $(F\phi)(b)$ be the pullback in the following diagram.

$$\begin{array}{ccc} (F\phi)(b) & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \phi \\ b\downarrow\mathcal{O} & \longrightarrow & \mathcal{O} \end{array}$$

Here, $b\downarrow\mathcal{O}$ is the usual comma category of objects under b . (The category $(F\phi)(b)$ is otherwise known as $b\downarrow\phi$.) If $\beta: b \rightarrow c$ is an arrow in \mathcal{O} , we have a functor $c\downarrow\mathcal{O} \rightarrow b\downarrow\mathcal{O}$ given by composition with β , and this makes $F\phi$ into a contravariant functor.

The right adjoint $E: \mathcal{O}\text{-Cat} \rightarrow \text{Cat}\downarrow\mathcal{O}$ is given by the Grothendieck construction [Gro71]. If $\psi: \mathcal{O} \rightarrow \text{Cat}$ is a contravariant functor, we define $E\psi: \bar{E}\psi \rightarrow \mathcal{O}$ as follows. The category $\bar{E}\psi$ has as set of objects the disjoint union of the sets of objects of all

of the categories $\psi(b)$; $E\psi$ maps each object in $\psi(b)$ to b . If $x \in \psi(b)$ and $y \in \psi(c)$, a map in $\bar{E}\psi$ from $x \rightarrow y$ is a pair (β, f) where $\beta: b \rightarrow c$ in \mathcal{O} and $f: x \rightarrow \beta^*y$ in $\psi(b)$. Composition of pairs is given by $(\beta, f) \circ (\gamma, g) = (\beta\gamma, \gamma^*f \circ g)$.

We can check directly that F and E are adjoint, or we can do the following. Note first that, if $\phi: \mathcal{C} \rightarrow \mathcal{O}$, then

$$\phi \cong \int^{\sigma \in \Delta \downarrow \mathcal{O}} \text{Hom}_{\mathcal{O}}(\sigma, \phi) \times \sigma$$

where $\Delta \downarrow \mathcal{O}$ is the full subcategory of $\text{Cat} \downarrow \mathcal{O}$ containing all maps $[n] \rightarrow \mathcal{O}$, and $\text{Hom}_{\mathcal{O}}$ denotes the set of maps over \mathcal{O} . (In fact, we could restrict to the subcategory of $\Delta \downarrow \mathcal{O}$ containing only those objects $[n] \rightarrow \mathcal{O}$ with $n \leq 2$.) It is easy to see that F preserves colimits, so we have that

$$F\phi \cong \int^{\sigma} \text{Hom}_{\mathcal{O}}(\sigma, \phi) \times F\sigma.$$

It follows from general nonsense about coends that F then has as its right adjoint the functor E defined by

$$E\psi = \int^{\sigma} \text{Hom}_{\mathcal{O}}(F\sigma, \psi) \times \sigma,$$

where $\text{Hom}_{\mathcal{O}}$ here denotes the set of natural transformations of functors defined on \mathcal{O} . Remembering that we need only consider $n \leq 2$, it's a pleasant exercise to see that E so defined can be described by the Grothendieck construction.

The constructions for simplicial sets are similar. The functor $F: \mathcal{S} \downarrow N\mathcal{O} \rightarrow \mathcal{O}\text{-}\mathcal{S}$ is defined as follows. Let $\phi: X \rightarrow N\mathcal{O}$ be a map of simplicial sets and let b be an object of \mathcal{O} . We let $(F\phi)(b)$ be the pullback in the following diagram.

$$\begin{array}{ccc} (F\phi)(b) & \longrightarrow & X \\ \downarrow & & \downarrow \phi \\ N(b \downarrow \mathcal{O}) & \longrightarrow & N\mathcal{O} \end{array}$$

We make $(F\phi)(b)$ a contravariant functor of b in the evident way. (Note: Since N preserves pullbacks, this diagram show that $FN = NF$.)

To define the right adjoint E , we first notice that F preserves colimits, as may be checked directly. Now, let $\phi: X \rightarrow N\mathcal{O}$ and $\sigma: \Delta[n] \rightarrow N\mathcal{O}$. (We shall freely move back and forth between such a map σ and the corresponding map $[n] \rightarrow \mathcal{O}$, which we shall also call σ .) Let $X_{\sigma} \subset X_n$ be the set of n -cells of X that map to $N\mathcal{O}$ via σ . Put another way, $X_{\sigma} = \text{Hom}_{N\mathcal{O}}(\sigma, \phi)$. We then have

$$\phi \cong \int^{\sigma \in \Delta \downarrow \mathcal{O}} X_{\sigma} \times \sigma.$$

(In fact, it is often convenient to think of a simplicial set over $N\mathcal{O}$ as instead a contravariant set-valued functor on $\Delta \downarrow \mathcal{O}$.) Using the fact that F preserves colimits,

we can then write

$$F\phi \cong \int^{\sigma} X_{\sigma} \times F\sigma.$$

It follows that the right adjoint E is given as follows. If $\psi: \mathcal{O} \rightarrow \mathcal{S}$ is a contravariant functor, then

$$E\psi = \int^{\sigma \in \Delta \downarrow \mathcal{O}} \text{Hom}_{\mathcal{O}}(F\sigma, \psi) \times \sigma,$$

so that

$$(E\psi)_{\sigma} = \text{Hom}_{\mathcal{O}}(F\sigma, \psi)$$

and

$$(E\psi)_n = \bigcup_{\sigma: \Delta[n] \rightarrow N\mathcal{O}} \text{Hom}_{\mathcal{O}}(F\sigma, \psi).$$

This is a generalization to simplicial sets of the Grothendieck construction for categories. As noted in [FG98], it is also the homotopy colimit of ψ .

Example 2.1. (See also Remark 3.10 in [Lat79].) Contrary to the claim in [FG98], if $\phi: \mathcal{C} \rightarrow \mathcal{O}$, $\psi: \mathcal{D} \rightarrow \mathcal{O}$, and $f: \phi \rightarrow \psi$ is a functor such that $f: \mathcal{C} \rightarrow \mathcal{D}$ is a weak equivalence of categories, then Ff need not be an objectwise weak equivalence. Consider the following example. Let $\mathcal{O} = [1]$, let $\mathcal{C} = [0]$ with $\phi: \mathcal{C} \rightarrow \mathcal{O}$ taking 0 to 0, and let $\mathcal{D} = [1]$ with $\psi: \mathcal{D} \rightarrow [1]$ the identity. Let $f: \phi \rightarrow \psi$ take 0 to 0. Then f is a weak equivalence of categories but $(F\phi)(1)$ is empty and $(F\psi)(1)$ is not, so Ff cannot be an objectwise weak equivalence. Taking the nerve gives an example showing that the similar claim about simplicial sets is also false. In fact, we shall see below that this example is prototypical of the cases that we need to avoid.

We now examine the fundamental diagrams $F\sigma$ in detail in the simplicial case. First, notice that $\sigma: [n] \rightarrow \mathcal{O}$ is determined by a sequence of maps

$$\sigma_0 \xrightarrow{g_1} \sigma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_n} \sigma_n$$

in \mathcal{O} . If b is an object of \mathcal{O} , we have the following pullback diagram.

$$\begin{array}{ccc} (F\sigma)(b) & \longrightarrow & \Delta[n] \\ \downarrow & & \downarrow \sigma \\ N(b \downarrow \mathcal{O}) & \longrightarrow & N\mathcal{O} \end{array}$$

Thus, a k simplex in $(F\sigma)(b)$ can be described as a pair (α, ω) where $\alpha: [k] \rightarrow [n]$ and $\omega: [k] \rightarrow b \downarrow \mathcal{O}$, with

$$\omega = (b \rightarrow \sigma\alpha(0) \xrightarrow{\sigma\alpha(0 \rightarrow 1)} \sigma\alpha(1) \xrightarrow{\sigma\alpha(1 \rightarrow 2)} \cdots \xrightarrow{\sigma\alpha(k-1 \rightarrow k)} \sigma\alpha(k)).$$

Write $\mathcal{O}(b, c)$ for the set of maps from b to c in \mathcal{O} and, for a fixed object c , write $\mathcal{O}(-, c)$ for the evident contravariant set-valued functor on \mathcal{O} . Define $\iota_n: \mathcal{O}(-, \sigma_0) \times \Delta[n] \rightarrow F\sigma$ to be the map taking $(g, \alpha) \in \mathcal{O}(b, \sigma_0) \times \Delta[n]_k$ to (α, ω) , where

$$\omega = (b \xrightarrow{\sigma(0 \rightarrow \alpha(0)) \circ g} \sigma\alpha(0) \xrightarrow{\sigma\alpha(0 \rightarrow 1)} \sigma\alpha(1) \xrightarrow{\sigma\alpha(1 \rightarrow 2)} \dots \xrightarrow{\sigma\alpha(k-1 \rightarrow k)} \sigma\alpha(k)).$$

Lemma 2.2. *Let $\sigma: \Delta[n] \rightarrow N\mathcal{O}$, let $A \subset \Delta[n]$ be a subsimplicial set containing the face opposite 0, and let λ denote the composite $A \rightarrow \Delta[n] \rightarrow N\mathcal{O}$. Then the following is a pushout diagram, where the top map is the restriction of ι_n .*

$$\begin{array}{ccc} \mathcal{O}(-, \sigma_0) \times A & \longrightarrow & F\lambda \\ \downarrow & & \downarrow \\ \mathcal{O}(-, \sigma_0) \times \Delta[n] & \xrightarrow{\iota_n} & F\sigma \end{array}$$

Proof. Let (α, ω) be an element of $(F\sigma)(b)$ for some object b . Then, $(\alpha, \omega) \in F\lambda$ if and only if $\alpha \in A$. Let us assume that $\alpha \notin A$. Since A contains the face opposite 0, we must have that $\alpha(0) = 0$. Then ω has the form

$$\omega = (b \xrightarrow{g} \sigma_0 \rightarrow c_1 \rightarrow \dots \rightarrow c_k),$$

and so (α, ω) is the image under ι_n of a unique element of $\mathcal{O}(b, \sigma_0) \times \Delta[n]$, i.e., (g, α) . Thus, $F\sigma$ is the pushout claimed. \square

We can now describe the structure of $F\sigma$.

Proposition 2.3. *If $\sigma: \Delta[0] \rightarrow N\mathcal{O}$, then $\iota_0: \mathcal{O}(-, \sigma_0) \times \Delta[0] \rightarrow F\sigma$ is an isomorphism. If $\sigma: \Delta[n] \rightarrow N\mathcal{O}$ with $n > 0$, we have the following pushout diagram.*

$$\begin{array}{ccc} \mathcal{O}(-, \sigma_0) \times \Delta[n-1] & \xrightarrow{\iota_{n-1} g_{1*}} & F(\sigma\delta_0) \\ \delta_0 \downarrow & & \downarrow \\ \mathcal{O}(-, \sigma_0) \times \Delta[n] & \xrightarrow{\iota_n} & F\sigma \end{array}$$

Here, $\delta_0: \Delta[n-1] \rightarrow \Delta[n]$ is inclusion of the face opposite 0.

Proof. This follows from Lemma 2.2 on letting A be the face opposite 0 ($A = \emptyset$ if $n = 0$). \square

This gives us a nice, Grothendieck-like interpretation of the right adjoint E . If $\psi: \mathcal{O} \rightarrow \mathcal{S}$ and $\sigma: \Delta[n] \rightarrow N\mathcal{O}$, the description of $F\sigma$ as a pushout allows us to describe $(E\psi)_\sigma = \text{Hom}_{\mathcal{O} \cdot \mathcal{S}}(F\sigma, \psi)$ as a pullback. Interpreting the terms involved, we get the following.

Corollary 2.4. *Let $\psi: \mathcal{O} \rightarrow \mathcal{S}$ and $\sigma: \Delta[n] \rightarrow N\mathcal{O}$, with*

$$\sigma = (\sigma_0 \xrightarrow{g_1} \dots \xrightarrow{g_n} \sigma_n).$$

Then an element of $(E\psi)_\sigma$ is specified by a sequence $(\tau_0, \tau_1, \dots, \tau_n)$ where $\tau_k \in \psi(\sigma_k)_{n-k}$ and $d_0\tau_{k-1} = g_k^*\tau_k$ for $1 \leq k \leq n$.

We can elaborate Example 2.1 to show that, if $\sigma: \Delta[n] \rightarrow N\mathcal{O}$ and λ is the composite $\Lambda^0[n] \rightarrow \Delta[n] \rightarrow N\mathcal{O}$, then $(F\lambda)(b) \rightarrow (F\sigma)(b)$ need not be a weak equivalence for every b . However, this is a problem for the 0th horn only.

Proposition 2.5. *Let $\sigma: \Delta[n] \rightarrow N\mathcal{O}$ and let λ be the composite $\Lambda^k[n] \rightarrow \Delta[n] \rightarrow N\mathcal{O}$ where $k > 0$. Then $(F\lambda)(b) \rightarrow (F\sigma)(b)$ is a weak equivalence for every object b in \mathcal{O} .*

Proof. Letting $A = \Lambda^k[n]$ in Lemma 2.2 (notice that this requires $k > 0$), we have the following pushout diagram.

$$\begin{array}{ccc} \mathcal{O}(b, \sigma_0) \times \Lambda^k[n] & \longrightarrow & (F\lambda)(b) \\ \downarrow & & \downarrow \\ \mathcal{O}(b, \sigma_0) \times \Delta[n] & \longrightarrow & (F\sigma)(b) \end{array}$$

But then standard results about simplicial sets say that $(F\lambda)(b) \rightarrow (F\sigma)(b)$ is a weak equivalence. \square

Another obvious application of Lemma 2.2 is to the case where $A = \partial\Delta[n]$. We shall use this later to examine the effect on cell structures of applying F .

3. THE EQUIVALENCES OF HOMOTOPY CATEGORIES

Although not usually stated explicitly, the following result or some variation on it is often used to show equivalences of homotopy categories when one category lacks a full model structure. Recall that, if \mathcal{C} is a category with certain of its maps designated as weak equivalences, then its homotopy category $h\mathcal{C}$ is the category obtained by inverting the weak equivalences; this category may or may not exist, the difficulty being a set-theoretic one we know is overcome for model categories.

Proposition 3.1. *Let \mathcal{C} be a category and let \mathcal{D} be a model category. Let $(L, R): \mathcal{C} \rightarrow \mathcal{D}$ be an adjoint pair such that the counit $\epsilon: LR(d) \rightarrow d$ is a weak equivalence for every object d in \mathcal{D} . Say that a map $f: x \rightarrow y$ in \mathcal{C} is a weak equivalence when Lf is one. Then the homotopy category of \mathcal{C} exists and is equivalent to the homotopy category of \mathcal{D} .*

The dual statement is the following.

Proposition 3.2. *Let \mathcal{C} be a model category and let \mathcal{D} be a category. Let $(L, R): \mathcal{C} \rightarrow \mathcal{D}$ be an adjoint pair such that the unit $\eta: c \rightarrow RL(c)$ is a weak equivalence for every object c in \mathcal{C} . Say that a map $f: x \rightarrow y$ in \mathcal{D} is a weak equivalence when Rf is one. Then the homotopy category of \mathcal{D} exists and is equivalent to the homotopy category of \mathcal{C} .*

The proofs of these results are straightforward. We can weaken the hypotheses: In Proposition 3.1 it suffices to assume that ϵ is a weak equivalence only on fibrant objects, and in Proposition 3.2 it suffices to assume that η is a weak equivalence only on cofibrant objects. Of course, if both categories are model categories then we can weaken the hypotheses further to the requirement that (L, R) be a Quillen equivalence.

In the context of Proposition 3.2, we can often push the conclusion further: If \mathcal{C} is cofibrantly generated and certain smallness assumptions hold, then L carries the generating cofibrations and acyclic cofibrations to cofibrant generators for a model category structure on \mathcal{D} . In that case (L, R) becomes a Quillen equivalence. Proposition 3.2 underlies the various proofs that the homotopy category of simplicial sets is equivalent to the homotopy category of small categories. Illusie [Ill72], Lee [Lee72], Latch [Lat77] and Thomason [Tho80] constructed various adjunctions $(L, R): \mathcal{S} \rightarrow \text{Cat}$ whose units are weak equivalences, with R naturally weakly equivalent to N . Any of these can be used to define a cofibrantly generated model category structure on Cat ; most of the work in [Tho80] (as corrected in [Cis99]) is to show that the structure constructed there is proper.

As mentioned in the introduction, it's well-known and not difficult to show that the category $\mathcal{O}\text{-}\mathcal{S}$ has a model category structure in which the weak equivalences are the objectwise weak equivalences and the fibrations are the objectwise fibrations. In fact, this structure is cofibrantly generated: For a set of generating cofibrations we may take the collection of maps $\mathcal{O}(-, c) \times \partial\Delta[n] \rightarrow \mathcal{O}(-, c) \times \Delta[n]$, and for a set of generating acyclic cofibrations we may take the collection of maps $\mathcal{O}(-, c) \times \Lambda^k[n] \rightarrow \mathcal{O}(-, c) \times \Delta[n]$, for all k and n .

Similarly, using the model category structure on Cat constructed by Thomason [Tho80], the category $\mathcal{O}\text{-Cat}$ has a cofibrantly generated model category structure in which the weak equivalences are the objectwise weak equivalences and the fibrations are the objectwise fibrations. The adjunction $(c\text{Sd}^2, \text{Ex}^2 N): \mathcal{S} \rightarrow \text{Cat}$ is a Quillen equivalence and is easily seen to induce a Quillen equivalence between $\mathcal{O}\text{-}\mathcal{S}$ and $\mathcal{O}\text{-Cat}$. Moreover, because $N \rightarrow \text{Ex}^2 N$ is a natural weak equivalence, we have the following result.

Theorem 3.3. *With weak equivalences defined objectwise, $N: \mathcal{O}\text{-Cat} \rightarrow \mathcal{O}\text{-}\mathcal{S}$ induces an equivalence of homotopy categories.*

We now define weak equivalence in $\text{Cat} \downarrow \mathcal{O}$ by saying that a map f over \mathcal{O} is a weak equivalence when Ff is one in $\mathcal{O}\text{-Cat}$. Similarly, we say that a map f in $\mathcal{S} \downarrow N\mathcal{O}$ is a weak equivalence when Ff is one in $\mathcal{O}\text{-}\mathcal{S}$. These are the notions of weak fibre homotopy equivalence from [Lat79].

To use Proposition 3.1, we need to show that the counits of the (F, E) adjunctions are weak equivalences, which we now do.

Proposition 3.4. *If $\psi: \mathcal{O} \rightarrow \text{Cat}$, then $\epsilon: FE\psi \rightarrow \psi$ is an objectwise weak equivalence.*

Proof. In fact, we shall show that, for every object b of \mathcal{O} , the functor $\epsilon: (FE\psi)(b) \rightarrow \psi(b)$ is a deformation retraction. Let $\mathcal{E} = (FE\psi)(b)$. Tracing through the definitions, we see that an object of \mathcal{E} is a pair (β, x) where $\beta: b \rightarrow c$ in \mathcal{O} and $x \in \psi(c)$. A map $(\beta, x) \rightarrow (\gamma: b \rightarrow d, y)$ is given by a pair (δ, ζ) where $\delta: c \rightarrow d$ with $\gamma = \delta\beta$ and $\zeta: x \rightarrow \delta^*y$. With this notation, ϵ is given by $\epsilon(\beta, x) = \beta^*x$ and $\epsilon(\delta, \zeta) = \beta^*\zeta$.

Now define $\xi: \psi(b) \rightarrow \mathcal{E}$ by $\xi(z) = (1_b, z)$ on objects and $\xi(\zeta) = (1_b, \zeta)$ on maps. We have $\epsilon\xi = 1$, and we can define a natural transformation $\omega: \xi\epsilon \rightarrow 1$ by $\omega(\beta, x) = (\beta, 1_{\beta^*x})$.

Thus, $\xi: \psi(b) \rightarrow (FE\psi)(b)$ is the inclusion of a deformation retract and ϵ is a weak equivalence. \square

Proposition 3.5. *If $\psi: \mathcal{O} \rightarrow \mathcal{S}$, then $\epsilon: FE\psi \rightarrow \psi$ is an objectwise weak equivalence.*

Proof. This statement is part of [FG98, 1.4], but we reprove it here. Let b be an object of \mathcal{O} . We begin by describing the simplicial set $(FE\psi)(b)$. We have the following pullback diagram.

$$\begin{array}{ccc} (FE\psi)(b) & \longrightarrow & \bar{E}\psi \\ \downarrow & & \downarrow \\ N(b\downarrow\mathcal{O}) & \longrightarrow & N\mathcal{O} \end{array}$$

Thus, an n -simplex of $(FE\psi)(b)$ is given by two compatible maps $\omega: \Delta[n] \rightarrow N(b\downarrow\mathcal{O})$ and $\alpha: \Delta[n] \rightarrow \bar{E}\psi$. The map ω is determined by a sequence

$$b \xrightarrow{g_0} \sigma_0 \xrightarrow{g_1} \dots \xrightarrow{g_n} \sigma_n$$

where $\sigma_0 \rightarrow \dots \rightarrow \sigma_n$ is the composite map $\sigma: \Delta[n] \rightarrow N\mathcal{O}$. The map α is then an element of $(E\psi)_\sigma$. By Corollary 2.4 we can describe such an element by a sequence $(\tau_0, \tau_1, \dots, \tau_n)$ where $\tau_k \in \psi(\sigma_k)_{n-k}$ and $d_0\tau_{k-1} = g_k^*\tau_k$ for $1 \leq k \leq n$. In summary, we can write an n -simplex of $(FE\psi)(b)$ as

$$(b \rightarrow \sigma_0 \rightarrow \dots \rightarrow \sigma_n; \tau_0, \dots, \tau_n)$$

with the compatibility conditions above.

The map $\epsilon: (FE\psi)(b) \rightarrow \psi(b)$ is then given by

$$\epsilon(b \xrightarrow{g_0} \sigma_0 \rightarrow \dots \rightarrow \sigma_n; \tau_0, \dots, \tau_n) = g_0^*(\tau_0).$$

We define $\xi: \psi(b) \rightarrow (FE\psi)(b)$ on an n -simplex τ by

$$\xi(\tau) = (b \rightarrow b \rightarrow \dots \rightarrow b; \tau_0, d_0\tau_0, \dots, d_0^n\tau_0)$$

where the maps $b \rightarrow b$ are all the identity. Clearly $\epsilon\xi = 1$, and it is straightforward to define a simplicial homotopy from $\xi\epsilon$ to the identity (the construction is very similar to standard arguments involving the bar construction). Thus ϵ is a weak homotopy equivalence. \square

From these two propositions and Proposition 3.1 we deduce the following.

Theorem 3.6. *With weak equivalence being weak fibre homotopy equivalence, $\text{Cat} \downarrow \mathcal{O}$ has a homotopy category, and $(F, E): \text{Cat} \downarrow \mathcal{O} \rightarrow \mathcal{O}\text{-Cat}$ induce inverse equivalences of homotopy categories. Similarly, $\mathcal{S} \downarrow N\mathcal{O}$ has a homotopy category and $(F, E): \mathcal{S} \downarrow N\mathcal{O} \rightarrow \mathcal{O}\text{-}\mathcal{S}$ induce inverse equivalences of homotopy categories.*

Combining this theorem with Theorem 3.3 we get the following (cf. [FG98, 1.5]).

Corollary 3.7. *Each functor in the following commutative diagram induces an equivalence of homotopy categories.*

$$\begin{array}{ccc} \text{Cat} \downarrow \mathcal{O} & \xrightarrow{F} & \mathcal{O}\text{-Cat} \\ N \downarrow & & \downarrow N \\ \mathcal{S} \downarrow N\mathcal{O} & \xrightarrow{F} & \mathcal{O}\text{-}\mathcal{S} \end{array}$$

So far we have avoided discussing model category structures for $\text{Cat} \downarrow \mathcal{O}$ and $\mathcal{S} \downarrow N\mathcal{O}$. In the remainder of this paper we construct a model category structure for $\mathcal{S} \downarrow N\mathcal{O}$ making $(F, E): \mathcal{S} \downarrow N\mathcal{O} \rightarrow \mathcal{O}\text{-}\mathcal{S}$ a Quillen equivalence. Given the difficult nature of Thomason's model category structure on Cat , and of any of the alternatives mentioned at the beginning of this section, we shall not here attempt to construct a model category structure on $\text{Cat} \downarrow \mathcal{O}$.

4. RIGHT FIBRATIONS

Before constructing the model category structure on $\mathcal{S} \downarrow N\mathcal{O}$ we need to examine a weakening of the idea of Kan fibration [Kan57] suggested by Proposition 2.5 and its failure in the case $k = 0$.

Definition 4.1. Let $J_r = \{\Lambda^k[n] \rightarrow \Delta[n] \mid n \geq 1 \text{ and } 0 < k \leq n\}$. A map of simplicial sets $p: X \rightarrow Y$ is a *right fibration* if it has the right lifting property (RLP) with respect to J_r (i.e., is in $J_r\text{-inj}$, using the notation of [Hov99]). A map $i: A \rightarrow B$ is a *right anodyne extension* if it has the left lifting property (LLP) with respect to all right fibrations (i.e., is in $J_r\text{-cof}$).

We could define analogous left fibrations and left anodyne extensions, using $J_l = \{\Lambda^k[n] \rightarrow \Delta[n] \mid n \geq 1 \text{ and } 0 \leq k < n\}$. However, we will have no use for these maps in this paper. A (Kan) fibration is a map that is both a right and a left fibration, so might be called a two-sided fibration.

Remark 4.2. The use of the terms “right” and “left” here is, of course, somewhat arbitrary. It is suggested by the translation to categories that occurs on applying categorical realization c (the left adjoint to the nerve). For categories, a functor $\phi: \mathcal{C} \rightarrow \mathcal{D}$ has the right lifting property with respect to $c\Lambda^2[2] \rightarrow [2]$ if the following is true: For every pair of maps $\alpha: x \rightarrow z$ and $\beta: y \rightarrow z$ in \mathcal{C} such that there is a

$g: \phi(x) \rightarrow \phi(y)$ such that $\phi(\alpha) = \phi(\beta) \circ g$, there exists a $\gamma: x \rightarrow y$ such that $\phi(\gamma) = g$ and $\alpha = \beta \circ \gamma$. We call this condition “right divisibility” (the right factor γ can be found). Similarly, having the right lifting property with respect to $c\Lambda^0[2] \rightarrow [2]$ we call left divisibility. A right fibration, therefore, has right divisibility but not necessarily left divisibility.

The results of [Hov99, §3.3] are true with anodyne extensions replaced by right anodyne extensions and fibrations replaced by right fibrations; Hovey’s proofs are easily modified for this case. In particular, the two main results of that section become the following.

Theorem 4.3. *If $i: K \rightarrow L$ is an inclusion of simplicial sets and $f: A \rightarrow B$ is a right anodyne extension, then*

$$i \sqcup f: (K \times B) \sqcup_{K \times A} (L \times A) \rightarrow L \times B$$

is a right anodyne extension.

Theorem 4.4. *If $i: K \rightarrow L$ is an inclusion of simplicial sets and $p: X \rightarrow Y$ is a right fibration, then*

$$\text{Map}_{\square}(i, p): \text{Map}(L, X) \rightarrow \text{Map}(K, X) \times_{\text{Map}(K, Y)} \text{Map}(L, Y)$$

is a right fibration.

Remark 4.5. It’s interesting that we can carry through the definition of homotopy groups as in [Hov99, §3.4] for right (or left) fibrant simplicial sets as easily as for fibrant ones (we just have to modify the proof that homotopy of vertices is an equivalence relation to avoid using the missing horns). By comparing the homotopy groups of X with those of ΩX (as in [Hov99, §3.6]) we can show that the homotopy groups we define this way agree with the usual ones (the homotopy groups of the geometric realization). However, we shall make no direct use of homotopy groups in this paper.

There is another construction that gives us right anodyne extensions. To describe it, we first discuss simplicial cones. Let Δ^- denote the category whose objects are the objects of Δ together with one extra object, the empty set. For notational convenience we write $[-1] = \emptyset$. Maps are again order-preserving functions, so $[-1]$ is initial in Δ^- . We can extend any simplicial set A naturally to a functor on Δ^- by letting $A_{-1} = *$, a one-point set. $\Delta[-1]$ is the empty simplicial set. (Δ^- is the category called Δ in [Mac71]; contravariant set-valued functors on Δ^- are there called augmented simplicial sets.) Let \mathcal{S}_* denote the category of based simplicial sets.

Definition 4.6. Let $C\Delta: \Delta^- \rightarrow \mathcal{S}_*$ be the covariant functor with $C\Delta[n] = \Delta[n+1]$ with 0 as the base vertex. If $f: [m] \rightarrow [n]$, let $\bar{f}: [m+1] \rightarrow [n+1]$ be the function defined by $\bar{f}(0) = 0$ and $\bar{f}(k) = f(k-1) + 1$ for $k > 0$; let $C\Delta(f) = \bar{f}^*$. Now extend

C to a functor $C: \mathcal{S} \rightarrow \mathcal{S}_*$ by letting

$$CA = \int^{n \in \Delta^-} A_n \times C\Delta[n].$$

The effect of C is to add an initial “cone point” to every simplex of A , these cone points all being identified to become the base vertex $*$ of CA . There is a natural inclusion $j: A \rightarrow CA$ (of unbased simplicial sets), induced by the map $\delta_0: \Delta[n] \rightarrow \Delta[n+1]$, the inclusion of the face opposite 0.

The description of the functor C as a coend shows that it has as its right adjoint $P: \mathcal{S}_* \rightarrow \mathcal{S}$, where

$$(PB)_n = \text{Hom}_{\mathcal{S}_*}(C\Delta[n], B) = \{z: \Delta[n+1] \rightarrow B \mid z(0) = *\}.$$

Dual to the inclusion j is the (unbased) map $\bar{j}: PB \rightarrow B$ that takes an $(n+1)$ -simplex in B to its face opposite 0. Although we’ve described PB as an unbased simplicial set, it has a canonical base vertex $*$, given by the map $C\Delta[0] \rightarrow B$ constant at the base vertex of B .

Theorem 4.7. *If $f: A \rightarrow B$ is an anodyne extension then $\hat{C}f: B \sqcup_A CA \rightarrow CB$ is a right anodyne extension.*

Proof. We first check the special case $f: \Lambda^k[n] \rightarrow \Delta[n]$. In this case, it is easy to see that $\hat{C}f$ is the inclusion $\Lambda^{k+1}[n+1] \rightarrow \Delta[n+1]$, which is a right anodyne extension since $k+1 > 0$. If we let

$$J = \{\Lambda^k[n] \rightarrow \Delta[n] \mid n \geq 1 \text{ and } 0 \leq k \leq n\},$$

and let $\hat{C}J$ denote the set that results when we apply \hat{C} to each map in J , then we have shown that every map in $\hat{C}J$ is a J_r -cofibration.

To get to the general case we note first that, by adjunction, liftings in a diagram as on the left below are in one-to-one correspondence with liftings in the diagram on the right.

$$\begin{array}{ccc} B \sqcup_A CA & \xrightarrow{\quad} & X \\ \hat{C}f \downarrow & \nearrow & \downarrow p \\ CB & \xrightarrow{\quad} & Y \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\quad} & PX \\ f \downarrow & \nearrow & \downarrow \hat{P}p \\ B & \xrightarrow{\quad} & PY \times_Y X \end{array}$$

In particular, if p has the RLP with respect to $\hat{C}J$, then $\hat{P}p$ has the RLP with respect to J (so is a Kan fibration).

Since every map in $\hat{C}J$ is a J_r -cofibration, every map in $\hat{C}J$ has the LLP with respect to J_r -inj, the set of J_r -injectives (the right fibrations). By adjunction, every map in J has the LLP with respect to $\hat{P}(J_r\text{-inj})$. It follows that every J -cofibration has the LLP with respect to $\hat{P}(J_r\text{-inj})$, hence, by adjunction, that every map in $\hat{C}(J\text{-cof})$ has the LLP with respect to $J_r\text{-inj}$. But this says that, if f is an anodyne extension, then $\hat{C}f$ is a right anodyne extension. \square

Cones are naturally contractible, in the following sense.

Lemma 4.8. *There is a natural map $H: C(A \times \Delta[1]) \rightarrow CA$ with the property that $H|_{C(A \times 0)}$ is constant at the base vertex while $H|_{C(A \times 1)}$ is the identity.*

Proof. We first define $H: C(\Delta[n] \times \Delta[1]) \rightarrow C\Delta[n]$. Recall that $\Delta[n] \times \Delta[1]$ has $n + 1$ top-dimensional nondegenerate simplices v_0 through v_n , where v_k is given by the sequences of vertices $(0, 0), (1, 0), \dots, (k, 0), (k, 1), \dots, (n, 1)$. Corresponding to these are $n + 1$ top-dimensional simplices of $C(\Delta[n] \times \Delta[1])$ we shall call w_0 through w_n . We define $H|_{w_k}: \Delta[n + 2] \rightarrow \Delta[n + 1]$ to be σ_0^{n+1-k} . We leave it to the reader to verify that this does define a simplicial map H with the stated properties, and that H is natural on Δ^- . We then extend H to all of \mathcal{S} using the coend:

$$C(A \times \Delta[1]) \cong \int^n A_n \times C(\Delta[n] \times \Delta[1]) \rightarrow \int^n A_n \times C\Delta[n] = CA.$$

□

By adjunction, we get a contraction of the dual construction.

Corollary 4.9. *There is a natural map $\hat{H}: PB \times \Delta[1] \rightarrow PB$ such that $\hat{H}|_{PB \times 0}$ is constant at the base vertex and $\hat{H}|_{PB \times 1}$ is the identity.*

A key technical result is the following.

Lemma 4.10. *Suppose that $p: X \rightarrow Y$ is a right fibration of simplicial sets. Suppose also that, for every vertex $v \in Y$, the fiber $p^{-1}(v)$ is an acyclic Kan complex. Then p is an acyclic fibration.*

Proof. We show that p has the RLP with respect to every map $\partial\Delta[n] \rightarrow \Delta[n]$. Consider the following lifting problem:

$$\begin{array}{ccc} \partial\Delta[n] & \xrightarrow{x} & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta[n] & \xrightarrow{y} & Y \end{array}$$

That this lifting problem can be solved for $n = 0$ is just the statement that the fibers, being acyclic, are nonempty. Let us now assume that $n > 1$ and think of $\partial\Delta[n] \rightarrow \Delta[n]$ as $\Delta[n - 1] \cup C\partial\Delta[n - 1] \rightarrow C\Delta[n - 1]$. We shall solve the lifting problem by first pulling it into a fiber, where the problem is solvable by assumption.

To do this, we consider first the following lifting problem:

$$\begin{array}{ccc} (\Delta[n - 1] \times 1) \cup C(\partial\Delta[n - 1] \times 1) & \xrightarrow{x} & X \\ \downarrow & \nearrow h & \downarrow p \\ (\Delta[n - 1] \times \Delta[1]) \cup C(\partial\Delta[n - 1] \times \Delta[1]) & \xrightarrow{yH} & Y \end{array}$$

The map yH is the restriction of the composite of y with the contraction $H: C(\Delta[n-1] \times \Delta[1]) \rightarrow C\Delta[n-1]$ of Lemma 4.8. We claim that the map on the left is a right anodyne extension. We see this by writing it as the composite

$$\begin{aligned} (\Delta[n-1] \times 1) \cup C(\partial\Delta[n-1] \times 1) &\rightarrow (\Delta[n-1] \times \Delta[1]) \cup C(\partial\Delta[n-1] \times 1) \\ &\rightarrow (\Delta[n-1] \times \Delta[1]) \cup C(\partial\Delta[n-1] \times \Delta[1]). \end{aligned}$$

The first of these maps is obtained by pushing out along $\Delta[n-1] \times 1 \rightarrow \Delta[n-1] \times \Delta[1]$, which is a right anodyne extension by Theorem 4.3. The second map is obtained by pushing out along

$$(\partial\Delta[n-1] \times \Delta[1]) \cup C(\partial\Delta[n-1] \times 1) \rightarrow C(\partial\Delta[n-1] \times \Delta[1]),$$

which is a right anodyne extension by Theorem 4.7. Therefore, the composite is a right anodyne extension as claimed, and so we can find the map h in the diagram above.

Now note that the image of $h|(\Delta[n-1] \times 0) \cup C(\partial\Delta[n-1] \times 0)$ lies entirely in a fiber of p . By assumption, then, we can extend h over $C(\Delta[n-1] \times 0)$. We now consider the following lifting problem:

$$\begin{array}{ccc} (\Delta[n-1] \times \Delta[1]) \cup C(\Delta[n-1] \times 0 \cup \partial\Delta[n-1] \times \Delta[1]) & \xrightarrow{h} & X \\ \downarrow & \nearrow k & \downarrow p \\ C(\Delta[n-1] \times \Delta[1]) & \xrightarrow{yH} & Y \end{array}$$

The map on the left is a right anodyne extension by Theorem 4.7, so we can find the lift k . The restriction of k to $C(\Delta[n-1] \times 1)$ is then the solution to our original lifting problem. \square

We also need to examine the relation between fibers and homotopy fibers.

Definition 4.11. If Y is a simplicial set and $y \in Y_0$ is a vertex, let (Y, y) denote the based simplicial set we get by considering y as the base vertex of Y . $P(Y, y)$ is then the simplicial set with

$$P(Y, y)_n = \{z: \Delta[n+1] \rightarrow Y \mid z(0) = y\}.$$

If $p: X \rightarrow Y$ is a map of simplicial sets and $y \in Y_0$ is a vertex, we define $P(p, y)$, the *(right) homotopy fiber over y* , to be the pullback in the following diagram:

$$\begin{array}{ccc} P(p, y) & \longrightarrow & X \\ \bar{p} \downarrow & & \downarrow p \\ P(Y, y) & \xrightarrow{\bar{y}} & Y \end{array}$$

When $Y = N\mathcal{O}$, $P(Y, y) = N(y\downarrow\mathcal{O})$ and so $P(p, y) = (Fp)(y)$, as defined in Section 2.

Recall that the fiber $p^{-1}(y)$ is the pullback in the following diagram:

$$\begin{array}{ccc} p^{-1}(y) & \longrightarrow & X \\ \downarrow & & \downarrow p \\ \Delta[0] & \xrightarrow{y} & Y \end{array}$$

The map $*$: $\Delta[0] \rightarrow P(Y, y)$ then induces a map $p^{-1}(y) \rightarrow P(p, y)$.

Proposition 4.12. *If $p: X \rightarrow Y$ is a right fibration of simplicial sets and y is a vertex of Y , then the map $p^{-1}(y) \rightarrow P(p, y)$ is the inclusion of a deformation retract.*

Proof. Since right lifting properties are preserved by pullback and $p: X \rightarrow Y$ is a right fibration, so is $\bar{p}: P(p, y) \rightarrow P(Y, y)$. Let $\hat{H}: P(Y, y) \times \Delta[1] \rightarrow P(Y, y)$ be the deformation constructed in Corollary 4.9 and consider the following lifting problem.

$$\begin{array}{ccc} p^{-1}(y) \times \Delta[1] \cup P(p, y) \times 1 & \longrightarrow & P(p, y) \\ \downarrow & \nearrow & \downarrow \bar{p} \\ P(p, y) \times \Delta[1] & \xrightarrow{\hat{H} \circ \bar{p}} & P(Y, y) \end{array}$$

Since $p^{-1}(y) \rightarrow P(p, y)$ is an inclusion, it follows from Theorem 4.3 that the map on the left is a right anodyne extension, hence we can solve this lifting problem. The lifted map is the desired deformation. \square

Remark 4.13. This result is clearly related to Proposition 3.5. However, it is not clear what set of assumptions would imply both results. Neither is it clear that such a unification would be useful here.

5. THE MODEL CATEGORY STRUCTURE AND THE QUILLEN EQUIVALENCE

We now define a new model category structure on $\mathcal{S}\downarrow N\mathcal{O}$.

Definition 5.1. Let $\phi: X \rightarrow N\mathcal{O}$ and $\psi: Y \rightarrow N\mathcal{O}$ be simplicial sets over $N\mathcal{O}$, and let $f: X \rightarrow Y$ be a map over $N\mathcal{O}$.

1. Say that f is a *weak equivalence over $N\mathcal{O}$* if, for each vertex $b \in N\mathcal{O}$, $(F\phi)(b) \rightarrow (F\psi)(b)$ is a weak equivalence of simplicial sets.
2. Say that f is a *cofibration over $N\mathcal{O}$* if it is a cofibration of simplicial sets, i.e., an inclusion.
3. Say that f is a *fibration over $N\mathcal{O}$* if it is a right fibration and if, for each vertex $b \in N\mathcal{O}$, $\phi^{-1}(b) \rightarrow \psi^{-1}(b)$ is a fibration of simplicial sets.

Theorem 5.2. *With the definitions of weak equivalence, cofibration, and fibration over $N\mathcal{O}$ given above, $\mathcal{S}\downarrow N\mathcal{O}$ is a cofibrantly generated model category. For a set of generating cofibrations we may take the set of all inclusions $\partial\Delta[n] \rightarrow \Delta[n]$ over $N\mathcal{O}$. For a set of generating acyclic cofibrations we may take the set of all inclusions $\Lambda^k[n] \rightarrow \Delta[n]$ over $N\mathcal{O}$ with $k > 0$, together with the inclusions $\Lambda^0[n] \rightarrow \Delta[n]$ in which the map $\Delta[n] \rightarrow N\mathcal{O}$ is constant at a vertex.*

Proof. We shall use the “recognition principle” [Hov99, 2.1.19] for cofibrantly generated categories. Hovey gives six criteria to check to verify that there is a cofibrantly generated model category structure with the stated weak equivalences, with the stated set I as a set of generating cofibrations and the stated set J as a set of generating acyclic cofibrations.

(1) *The collection of weak equivalences has the two-out-of-three property and is closed under retracts.* This is straightforward to check given that it is true for ordinary weak equivalences of simplicial sets.

(2) & (3) *The domains of I and J are small.* This follows from the fact that all simplicial sets are small.

(4) *Relative J -cell complexes are acyclic I -cofibrations.* Every map in J is an inclusion, from which it follows that relative J -cell complexes are I -cofibrations. It suffices, then, to check that every map in J is a weak equivalence over $N\mathcal{O}$. But this is the content of Proposition 2.5. (The case $\Lambda^0[n] \rightarrow \Delta[n] \rightarrow \text{vertex}$ is easy.)

(5) *I -injectives are weak equivalences.* Let $\phi: X \rightarrow N\mathcal{O}$ and $\psi: Y \rightarrow N\mathcal{O}$, and let $p: X \rightarrow Y$ be an I -injective over $N\mathcal{O}$. Then p is an acyclic Kan fibration. If b is any vertex in $N\mathcal{O}$, a diagram chase shows that the following is a pullback diagram.

$$\begin{array}{ccc} (F\phi)(b) & \longrightarrow & X \\ \downarrow & & \downarrow p \\ (F\psi)(b) & \longrightarrow & Y \end{array}$$

It follows that $(F\phi)(b) \rightarrow (F\psi)(b)$ is an acyclic Kan fibration for every b , hence that $X \rightarrow Y$ is a weak equivalence over $N\mathcal{O}$.

(6) *Acyclic J -injectives are I -injectives.* Let $\phi: X \rightarrow N\mathcal{O}$ and $\psi: Y \rightarrow N\mathcal{O}$, and let $p: X \rightarrow Y$ be an acyclic J -injective over $N\mathcal{O}$. By the inclusion in J of the maps $\Lambda^0[n] \rightarrow \Delta[n] \rightarrow \text{vertex}$, it follows that $\phi^{-1}(b) \rightarrow \psi^{-1}(b)$ is a Kan fibration for every vertex b in $N\mathcal{O}$. On the other hand, we have the following commutative diagram, in which the horizontal maps are inclusions of deformation retracts (by Proposition 4.12) and the map on the right is a weak equivalence by assumption.

$$\begin{array}{ccc} \phi^{-1}(b) & \longrightarrow & (F\phi)(b) \\ \downarrow & & \downarrow \\ \psi^{-1}(b) & \longrightarrow & (F\psi)(b) \end{array}$$

It follows that $\phi^{-1}(b) \rightarrow \psi^{-1}(b)$ is a weak equivalence, hence an acyclic Kan fibration, for each b . From this we can conclude that, for each vertex y of Y , $p^{-1}(y)$ is an acyclic Kan complex. We can now appeal to Lemma 4.10 to conclude that p is an acyclic Kan fibration, hence is an I -injective.

Thus, by [Hov99, 2.1.19], $\mathcal{S}\downarrow N\mathcal{O}$ is a cofibrantly generated model category with the stated weak equivalences and generating cofibrations. What remains to show is the characterizations of cofibrations and fibrations over $N\mathcal{O}$. But, it is obvious that the I -cofibrations are just the usual cofibrations of simplicial sets, i.e., inclusions. It is also clear that the J -injectives are precisely the right fibrations for which each map of fibers $\phi^{-1}(b) \rightarrow \psi^{-1}(b)$ is a Kan fibration. \square

Note that we can also characterize acyclic fibrations over $N\mathcal{O}$: They are precisely the usual acyclic Kan fibrations because this is the class that has the RLP with respect to I .

Now we want to compare this model category to $\mathcal{O}\text{-}\mathcal{S}$. As mentioned earlier, the model category structure on $\mathcal{O}\text{-}\mathcal{S}$ is cofibrantly generated; for a set of generating cofibrations we may take the collection of maps $\mathcal{O}(-, c) \times \partial\Delta[n] \rightarrow \mathcal{O}(-, c) \times \Delta[n]$, and for a set of generating acyclic cofibrations we may take the collection of maps $\mathcal{O}(-, c) \times \Lambda^k[n] \rightarrow \mathcal{O}(-, c) \times \Delta[n]$, for all k and n .

Theorem 5.3. *Let $\mathcal{S}\downarrow N\mathcal{O}$ and $\mathcal{O}\text{-}\mathcal{S}$ have the model category structures described above. Then $(F, E): \mathcal{S}\downarrow N\mathcal{O} \rightarrow \mathcal{O}\text{-}\mathcal{S}$ is a Quillen equivalence.*

Proof. We show first that F preserves cofibrations and acyclic cofibrations. To show that F preserves cofibrations, it suffices to show that it takes each generating cofibration to a cofibration. But, Lemma 2.2 shows that, if $\sigma: \Delta[n] \rightarrow N\mathcal{O}$ and λ is the composite $\partial\Delta[n] \rightarrow \Delta[n] \rightarrow N\mathcal{O}$, we have the following pushout diagram.

$$\begin{array}{ccc} \mathcal{O}(-, \sigma_0) \times \partial\Delta[n] & \longrightarrow & F\lambda \\ \downarrow & & \downarrow \\ \mathcal{O}(-, \sigma_0) \times \Delta[n] & \longrightarrow & F\sigma \end{array}$$

From this it is clear that $F\lambda \rightarrow F\sigma$ is a cofibration, so F preserves cofibrations. On the other hand, F preserves weak equivalences by definition, so F also preserves acyclic cofibrations. Therefore, (F, E) is a Quillen adjunction.

To show that (F, E) is a Quillen equivalence, we need to show that, if $\phi: A \rightarrow N\mathcal{O}$ is cofibrant (which is always true) and $\psi: \mathcal{O} \rightarrow \mathcal{S}$ is fibrant, then a map $f: \phi \rightarrow E\psi$ is a weak equivalence if and only if its adjoint $\hat{f}: F\phi \rightarrow \psi$ is. Since f is a weak equivalence if and only if Ff is, it suffices to show that the counit $\epsilon: FE\psi \rightarrow \psi$ is a weak equivalence, but this was done in Proposition 3.5. \square

In fact, $\mathcal{S}\downarrow N\mathcal{O}$ and $\mathcal{O}\text{-}\mathcal{S}$ are both simplicial model categories and (F, E) is a Quillen equivalence of simplicial model categories, as can be easily checked.

REFERENCES

- [Cis99] D.-C. Cisinski, *La classe des morphismes de Dwyer n'est pas stable par retracts*, Cahiers Topologie Géom. Différentielle Catégoriques **40** (1999), 227–231.
- [FG98] R. Fritsch and M. Golasinski, *Simplicial and categorical diagrams, and their equivariant applications*, Theory Appl. Categories **4** (1998), 73–81.
- [Gro71] A. Grothendieck, *Révetements étales et groupe fondamental*, Lecture Notes in Math., vol. 224, Springer-Verlag, 1971.
- [Hov99] M. Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, Amer. Math. Soc., 1999.
- [Ill72] L. Illusie, *Complex cotangent et déformations II*, Lecture Notes in Math., vol. 283, Springer Verlag, 1972.
- [Kan57] D. M. Kan, *On c. s. s. complexes*, Amer. J. Math. **79** (1957), 449–476.
- [Lat77] D. M. Latch, *The uniqueness of homology for the category of small categories*, J. Pure Appl. Algebra **9** (1977), 221–237.
- [Lat79] ———, *A fibred homotopy equivalence and homology theories for the category of small categories*, J. Pure Appl. Algebra **15** (1979), 247–269.
- [Lee72] M.-J. Lee, *Homotopy for functors*, Proc. Amer. Math. Soc. **36** (1972), 571–577.
- [Mac71] S. Mac Lane, *Categories for the working mathematician*, Springer-Verlag, 1971.
- [Tho80] R. W. Thomason, *Cat as a closed model category*, Cahiers Topologie Géom. Différentielle **XXI** (1980), 305–324.

DEPARTMENT OF MATHEMATICS, 103 HOFSTRA UNIVERSITY, HEMPSTEAD, NY 11549-1030
E-mail address: Steven.R.Costenoble@Hofstra.edu